

Geometry of infinite dimensional Grassmannians and the Mickelsson-Rajeev cocycle

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Abstract

In their study of the representation theory of loop groups, Pressley and Segal introduced a determinant line bundle over an infinite dimensional Grassmann manifold. Mickelsson and Rajeev subsequently generalized the work of Pressley and Segal to obtain representations of the groups $\text{Map}(M, G)$ where M is an odd dimensional spin manifold. In the course of their work, Mickelsson and Rajeev introduced for any $p \geq 1$, an infinite dimensional Grassmannian Gr_p and a determinant line bundle Det_p over it, generalizing the constructions of Pressley and Segal. The definition of the line bundle Det_p requires the notion of a regularized determinant for bounded operators. In this note we specialize to the case when $p = 2$ (which is relevant for the case when $\dim M = 3$) and consider the geometry of the determinant line bundle Det_2 . We construct explicitly a connection on Det_2 and give a simple formula for its curvature. From our results we obtain a geometric derivation of the Mickelsson-Rajeev cocycle.

1 Introduction

In the paper [8], the authors construct representations of the groups $\text{Map}(M, G)$, generalizing the methods of Pressley and Segal [11] for constructing representations of loop groups. Here M is a compact spin manifold of odd dimension and G is a compact Lie group. In the work of Pressley and Segal a fundamental role was played by the *restricted general linear group* GL_{res} and the *restricted Grassmannian* Gr_{res} associated to a polarized Hilbert space $H = H_+ \oplus H_-$. GL_{res} and Gr_{res} were defined relative to a certain Schatten ideal, namely the Hilbert-Schmidt operators. Recall that for any $p \geq 1$ one can define ideals L_p — the *Schatten ideals* — in the space $\mathcal{B}(H)$ of bounded operators on H (see for example [14]). When the Schatten index $p = 1$, the ideal L_1 is just the ideal of trace class operators on H , and when $p = 2$ the ideal L_2 is the ideal of Hilbert-Schmidt operators on H , as we have mentioned. These Schatten ideals play an

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important role in non-commutative geometry [3]. They arise also in the work of Mickelsson and Rajeev. In [8] the group GL_{res} was generalized to the group $GL_{(p)}$, for any Schatten index $p \geq 1$. The group $GL_{(p)}$ (not to be confused with the general linear group of a p -dimensional vector space!) was defined to be the group of invertible operators g on H such that with respect to the polarization $H = H_+ \oplus H_-$ the operator g has 2×2 block operator form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in which the off diagonal blocks b and c belong to the Schatten class L_{2p} . Gr_p is the associated Grassmannian, again defined relative to L_{2p} . We shall recall the definition of Gr_p in greater detail in Section 3. In the framework of Mickelsson and Rajeev GL_{res} and Gr_{res} correspond to $GL_{(1)}$ and Gr_1 respectively. The Schatten index p arises in [8] in the following way. In that paper it is shown that there is an embedding of the group $\text{Map}(M, G)$ into the general linear group $GL_{(p)}$ of a certain polarized Hilbert space provided that p exceeds a certain bound, related to the dimension of M . This generalizes the embedding defined by Pressley and Segal of the loop group $\text{Map}(S^1, G)$ into GL_{res} .

In the case of the ordinary Grassmannian $\text{Gr}(V)$ associated to a finite dimensional vector space V , there is a canonical holomorphic determinant line bundle Det defined over $\text{Gr}(V)$. If W is a subspace of V belonging to some connected component of $\text{Gr}(V)$ then the fiber of Det at W is the top exterior power $\Lambda^{\text{top}} W$. In the case of the infinite dimensional Grassmannian Gr_1 the notion of the top exterior power loses its meaning. Nevertheless, Pressley and Segal construct a well defined holomorphic determinant line bundle Det on Gr_{res} , and moreover show that there is a central extension of groups

$$1 \rightarrow \mathbb{C}^* \rightarrow \widehat{GL}_{\text{res}} \rightarrow GL_{\text{res}} \rightarrow 1 \quad (1)$$

with the property that the group $\widehat{GL}_{\text{res}}$ acts on the space Γ of holomorphic sections of the dual bundle Det^* . Pressley and Segal show that Γ can be interpreted as the fermionic Fock space construction on H . Inside GL_{res} is the subgroup U_{res} consisting of all unitary operators in GL_{res} . Corresponding to U_{res} is a subgroup \widehat{U}_{res} of $\widehat{GL}_{\text{res}}$ and it turns out that the irreducible representation of $\widehat{GL}_{\text{res}}$ on Γ restricts to an irreducible unitary representation of \widehat{U}_{res} . This irreducible representation of \widehat{U}_{res} is used to construct the ‘basic’ positive energy representation of the loop group LU_n using an embedding $LU_n \subset U_{\text{res}}$. This is the construction the authors in [8] worked towards generalizing to the groups $\text{Map}(M, G)$.

Using a notion of regularized determinant [14] for invertible operators in $1 + L_p$ Mickelsson and Rajeev construct *regularized determinant line bundles* Det_p on Gr_p . The central extension (1) above is replaced by an extension

$$1 \rightarrow \text{Map}(\text{Gr}_p, \mathbb{C}^*) \rightarrow \widehat{GL}_{(p)} \rightarrow GL_{(p)} \rightarrow 1 \quad (2)$$

which is now non-central. An extension of $\widehat{GL}_{(p)}$ is obtained on the space of smooth sections of Det_p^* . This extension satisfied a positive energy condition,

however it was later shown [7, 10] that this was not a unitary extension when $p = 2$. As mentioned above the appearance of the Schatten indices p in the work of Mickelsson and Rajeev arises from an embedding $\text{Map}(M, G) \subset GL_{(p)}$, where p depends on the dimension of M . The case when the dimension of M is 3 corresponds to $p = 2$.

The purpose of this note is to study the geometry of the Grassmannian Gr_2 . Our main result (Proposition 3) gives an explicit construction of a connection on Det_2 and an explicit and simple formula for the corresponding curvature 2-form. We point out that this is not as trivial as it seems. It is easy to do this for the determinant line bundle over Gr_{res} studied by Pressley and Segal, but the case of Det_2 is much more delicate. The same difficulties arise in finding closed formulas for certain universal Schwinger cocycles: when $p = 1$ there is the well known Kac-Peterson cocycle (see [11] Proposition 6.6.5), when $p = 2$ a considerably more difficult calculation produces the Mickelsson-Rajeev cocycle (see [8] and the discussion below). No closed formula is known for these universal cocycles for arbitrary p (but see [4] for a conjectural formula).

We should also mention that the line bundles Det_p are not just holomorphic line bundles, but they also carry a Hermitian structure as well. Therefore there is a canonical connection on each Det_p compatible with both the Hermitian and holomorphic structures. There is a formula for the curvature of this canonical connection however it appears to be quite difficult to derive a simple expression for it, at least for $p > 1$. This canonical connection featured also in Quillen's paper [13] however there he managed to identify its curvature with a certain Kähler form.

One consequence of our result is that we obtain a simple and explicit formula for a de Rham representative of the first Chern class $c_1(\text{Gr}_2)$ of Gr_2 . Recall that the topology of the Grassmannians Gr_p is well understood [9, 12]. For any $p < q$ there is a natural inclusion $\text{Gr}_p \subset \text{Gr}_q$ and this turns out to be a homotopy equivalence. The infinite dimensional manifolds Gr_p give smooth models for the classifying space of even K -theory K^0 and in fact it turns out that the de Rham theorem holds for them. This was exploited by Quillen in [12] where he gave explicit formulas in terms of contour integrals for differential form representatives of the Chern classes $c_n(\text{Gr}_p)$ for all n and p . The expressions he obtained however were not easy to evaluate directly.

Another by-product of our construction of a connection on Det_2 is that we are able to give a geometric derivation of the *Mickelsson-Rajeev cocycle* associated to the extension of Lie algebras

$$0 \rightarrow \text{Map}(\text{Gr}_2, \mathbb{C}) \rightarrow \widehat{\mathfrak{gl}}_{(2)} \rightarrow \mathfrak{gl}_{(2)} \rightarrow 0 \quad (3)$$

which is the infinitesimal version of (2) (again note the potential confusion with the finite dimensional Lie algebra $\mathfrak{gl}(2)!$). This cocycle was derived in [8] however the computation used there was rather involved. An alternative, algebraic derivation can be found in [5]. Using the curvature 2-form of our connection on Det_2 we give an alternative expression for the cocycle associated to the extension (3). This does not agree on the nose with the cocycle obtained

by Mickelsson and Rajeev, but we give an explicit formula for a coboundary relating the two cocycles.

The extension (2) is an example of an extension of groups that arises naturally whenever one has a line bundle L on a manifold M on which a Lie group G acts. In this situation there is a canonical extension

$$1 \rightarrow \text{Map}(M, \mathbb{C}^*) \rightarrow \widehat{G} \rightarrow G \rightarrow 1 \quad (4)$$

of G by the abelian group $\text{Map}(M, \mathbb{C}^*)$. The Mickelsson-Rajeev extension is a special case of this canonical extension. We describe two methods for associating a Lie algebra 2-cocycle to the extension of Lie algebras associated to (4). The first method is geometric, using a connection on L . The second method is more algebraic, requiring a knowledge of the local structure of (4). The two methods lead to different 2-cocycles in general. We give a formula, which appears to be new, for a coboundary relating these 2-cocycles.

In summary then this paper is as follows. In Section 2 we discuss the two methods for associating a cocycle to the extension of Lie algebras associated to (4) and derive a formula for a coboundary relating the two different cocycles we obtain. In Sections 3 and 4 we provide some background on the infinite dimensional Grassmann manifold Gr_2 and the determinant line bundle Det_2 . In Section 5 we construct a connection 1-form on Det_2 and compute its curvature. Section 6 contains a comparison of the geometric cocycle describing the extension (3) that we obtain from the curvature with the Mickelsson-Rajeev cocycle. Two slightly complicated calculations are contained in the appendices.

2 General remarks on Lie algebra 2-cocycles

Suppose that M is a \mathfrak{g} -module for a Lie algebra \mathfrak{g} . We can consider M as an abelian Lie algebra and consider extensions of Lie algebras

$$0 \rightarrow M \rightarrow \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \rightarrow 0.$$

If the linear map underlying the homomorphism p admits a section (so that $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus M$ as a vector space) then one can associate to the extension a Lie algebra 2-cocycle ω with values in the \mathfrak{g} -module M ; thus ω is a linear map $\omega: \Lambda^2 \mathfrak{g} \rightarrow M$ such that

$$\begin{aligned} \omega([\xi, \eta], \zeta) - \omega([\xi, \zeta], \eta) + \omega([\eta, \zeta], \xi) \\ - \xi \cdot \omega(\eta, \zeta) + \eta \cdot \omega(\xi, \zeta) - \zeta \cdot \omega(\xi, \eta) = 0 \end{aligned} \quad (5)$$

Conversely, given such a cocycle one may use it to twist the Lie bracket on $\mathfrak{g} \oplus M$ to obtain a new Lie algebra $\widehat{\mathfrak{g}}$ fitting into an extension of Lie algebras as above. This is a brief summary of the well-known theorem that isomorphism classes of extensions of \mathfrak{g} by the abelian Lie algebra M are classified by the Lie algebra cohomology group $H^2(\mathfrak{g}, M)$.

A nice example of such an extension of Lie algebras arises as the infinitesimal version of the extension of Lie groups (4) mentioned in the Introduction. Suppose L is a line bundle on a manifold M on which a Lie group G acts. Then there is canonically associated to G an extension of Lie groups

$$1 \rightarrow \text{Map}(M, \mathbb{C}^*) \rightarrow \widehat{G} \rightarrow G \rightarrow 1 \quad (6)$$

where \widehat{G} is the subgroup of the group of bundle automorphisms of L consisting of automorphisms which cover the action of G on M . There is a corresponding infinitesimal version of this; if \mathfrak{g} denotes the Lie algebra of G then we have the extension of Lie algebras

$$0 \rightarrow \text{Map}(M, \mathbb{C}) \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0. \quad (7)$$

Here a vector $\xi \in \mathfrak{g}$ acts on a function $f \in \text{Map}(M, \mathbb{C})$ by

$$(\xi \cdot f)(x) = \left. \frac{d}{dt} \right|_{t=0} f(x \exp(t\xi)). \quad (8)$$

In fact every vector $\xi \in \mathfrak{g}$ generates a vector field $\hat{\xi}$ on M , the fundamental vector field generated by the infinitesimal action of ξ . The value of $\hat{\xi}$ on a function f on M is given by precisely the same derivative formula as in (8).

It is important to note that in general the extensions (6) and (7) are not central extensions. We will call such an extension of groups by an abelian normal subgroup an *abelian extension* of groups, and we will also call an extension of Lie algebras by an abelian ideal, an *abelian extension* of Lie algebras.

By the classification theorem mentioned above, the abelian extension of Lie algebras (7) will be described by a Lie algebra 2-cocycle ω on \mathfrak{g} with values in the \mathfrak{g} -module $\text{Map}(M, \mathbb{C})$. Since the extension of groups, and hence the associated extension (7), is completely determined by the line bundle L on M and the action of the group G on M , then one should expect to find a formula for the cocycle ω in terms of some geometric data on L . Indeed, if L comes equipped with a connection ∇ whose curvature 2-form is F_∇ then one can describe ω as follows:

$$\omega(\xi, \eta) = -F_\nabla(\hat{\xi}, \hat{\eta}) \quad (9)$$

where $\hat{\xi}$ and $\hat{\eta}$ are the fundamental vector fields on M generated by the infinitesimal action of $\xi, \eta \in \mathfrak{g}$. The condition (5) that ω is a Lie algebra 2-cocycle is exactly the condition that the curvature F_∇ is a closed 2-form on M .

Another method to compute the cocycle ω is to use the local structure of the group \widehat{G} . Here it is important to realise that *locally* the underlying manifold of the Lie group \widehat{G} is a product of G and $\text{Map}(M, \mathbb{C}^*)$ but this is not in general true *globally*. In other words \widehat{G} is a locally trivial principal $\text{Map}(M, \mathbb{C}^*)$ bundle over G . In this method one chooses a local section σ defined in a neighbourhood of 1 of the map underlying the homomorphism $\widehat{G} \rightarrow G$ and defines a 2-cocycle $\omega(\xi, \eta)$ by

$$\left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} \sigma(e^{t\xi}) \sigma(e^{s\eta}) \sigma(e^{-t\xi}) \sigma(e^{-s\eta}) = ([\xi, \eta], \omega(\xi, \eta)) \quad (10)$$

where $\exp(t\xi)$ and $\exp(s\eta)$ are 1-parameter subgroups. Here the right hand side should be understood in terms of the splitting $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus \text{Map}(M, \mathbb{C})$ of the exact sequence (7) of Lie algebras defined by $d\sigma$, the derivative of the local section σ at the identity. Thus another way to think of $\omega(\xi, \eta)$ is as the familiar expression

$$\omega(\xi, \eta) = [d\sigma(\xi), d\sigma(\eta)] - d\sigma[\xi, \eta].$$

We remark that the expression (10) can be difficult to evaluate, cf. equation (4.12) in [8].

We have then two different descriptions of the Lie algebra 2-cocycle associated to the extension (7) — let us denote the 2-cocycle obtained from the geometric data (i.e the curvature) by $\omega_G(\xi, \eta)$ and the 2-cocycle obtained from the local, algebraic structure by $\omega_A(\xi, \eta)$. Since the cocycles $\omega_G(\xi, \eta)$ and $\omega_A(\xi, \eta)$ define the same extension of Lie algebras they should be cohomologous. In fact one can write down an explicit formula for a coboundary¹ relating them, this is the content of our first proposition.

Before we state the proposition, we will make a remark about line bundles. Throughout the paper we will blur the distinction between line bundles and principal \mathbb{C}^* bundles. To every line bundle L is associated a principal \mathbb{C}^* bundle L^+ , its principal frame bundle. The association of L^+ to L sets up an equivalence of categories between the category of line bundles on M and the category of principal \mathbb{C}^* bundles on M . It is well known (see for example [1]) that this equivalence between line bundles and principal \mathbb{C}^* bundles extends to connections: to every connection ∇ on a line bundle L there is associated a connection 1-form A on the principal \mathbb{C}^* bundle L^+ and conversely. For more details we refer to [1]. With these remarks made, we will freely pass between line bundles and principal \mathbb{C}^* bundles without further comment.

We recall that L was a line bundle over the G -manifold M equipped with a connection ∇ . We will denote by A the corresponding connection 1-form and we will denote by σ a local section of $\widehat{G} \rightarrow G$ defined in a neighborhood of the identity.

Proposition 1. *Let L , M , G , A and σ be as above. Then the two Lie algebra cocycles $\omega_G(\xi, \eta)$ and $\omega_A(\xi, \eta)$ are related by the coboundary $b(\xi)$ defined by*

$$b(\xi) = A(\widehat{d\sigma(\xi)}). \quad (11)$$

In other words we have

$$\omega_A(\xi, \eta) = \omega_G(\xi, \eta) + \xi \cdot b(\eta) - \eta \cdot b(\xi) - b([\xi, \eta]).$$

First suppose that ξ is a vector in \mathfrak{g} . Then $d\sigma(\xi)$ is a vector in $\widehat{\mathfrak{g}}$ and we can consider the fundamental vector field $\widehat{d\sigma(\xi)}$ on L induced by the infinitesimal action of $d\sigma(\xi)$. We can also consider the horizontal lift $(\hat{\xi})_H$ of the fundamental

¹We haven't been able to find a reference in which the formula above for the coboundary is described, but we would be surprised if it were not known.

vector field $\hat{\xi}$ on M . The vector field $\widehat{d\sigma(X)} - (\hat{\xi})_H$ on L is vertical with respect to the \mathbb{C}^* action on L . Consider

$$\begin{aligned} [\widehat{d\sigma(\xi)}, \widehat{d\sigma(\eta)}] &= [\widehat{d\sigma(\xi)} - (\hat{\xi})_H, \widehat{d\sigma(\eta)} - (\hat{\eta})_H] + [(\hat{\xi})_H, \widehat{d\sigma(\eta)} - (\hat{\eta})_H] \\ &\quad + [\widehat{d\sigma(\xi)} - (\hat{\xi})_H, (\hat{\eta})_H] + [(\hat{\xi})_H, (\hat{\eta})_H] \end{aligned}$$

The term $[\widehat{d\sigma(\xi)} - (\hat{\xi})_H, \widehat{d\sigma(\eta)} - (\hat{\eta})_H]$ in this expression vanishes, since it is a bracket of the form $[\hat{\alpha}, \hat{\beta}]$ where α and β are complex numbers. We have, since $F_A(\hat{\xi}, \hat{\eta}) = -A([\hat{\xi}]_H, (\hat{\eta})_H]$,

$$A([\widehat{d\sigma(\xi)}, \widehat{d\sigma(\eta)}]) = A([\widehat{d\sigma(\xi)} - (\hat{\xi})_H, (\hat{\eta})_H]) + A([\hat{\xi}]_H, \widehat{d\sigma(\eta)} - (\hat{\eta})_H]) - F_A(\hat{\xi}, \hat{\eta}).$$

If V and W are vectors in $\text{Lie}(\widehat{G})$ then we have the relation $[\widehat{V}, \widehat{W}] = [\widehat{V}, \widehat{W}]$ between fundamental vector fields. It follows therefore that we have

$$\begin{aligned} c(\xi, \eta) &= A([\widehat{d\sigma(\xi)}, \widehat{d\sigma(\eta)}]) - A(\widehat{d\sigma[\xi, \eta]}) \\ &= -F_A(\hat{\xi}, \hat{\eta}) + A([\hat{\xi}]_H, \widehat{d\sigma(\eta)} - (\hat{\eta})_H]) - A([\hat{\eta}]_H, \widehat{d\sigma(\xi)} - (\hat{\xi})_H]) \\ &\quad - A(\widehat{d\sigma[\xi, \eta]}) \end{aligned}$$

To simplify the terms $A([\hat{\xi}]_H, \widehat{d\sigma(\eta)} - (\hat{\eta})_H])$ and $A([\hat{\eta}]_H, \widehat{d\sigma(\xi)} - (\hat{\xi})_H])$ consider more generally $A([X, Y])$ where the vector field X is vertical and the vector field Y is horizontal. Then it is easy to see that we have $A([X, Y]) = -\mathcal{L}_Y A(X)$. Therefore we can write

$$A([\hat{\xi}]_H, \widehat{d\sigma(\eta)} - (\hat{\eta})_H]) = \mathcal{L}_{(\hat{\xi})_H} A(\widehat{d\sigma(\eta)} - (\hat{\eta})_H) = \mathcal{L}_{(\hat{\xi})_H} A(\widehat{d\sigma(\eta)}).$$

The vector field $\widehat{d\sigma(\eta)}$ on L is invariant under the action of \mathbb{C}^* : $dR_z(\widehat{d\sigma(\eta)})_p = \widehat{d\sigma(\eta)}_{pz}$, and so $A(\widehat{d\sigma(\eta)})$ descends to a function on M , which we will continue to denote $A(\widehat{d\sigma(\eta)})$. If V is a vector field on M and f is a function on M then we calculate the Lie derivative $\mathcal{L}_V(f)$ at $x \in M$ by choosing a path $\gamma: (-\epsilon, \epsilon) \rightarrow M$ through x with $\gamma'(0) = V$ and taking the derivative

$$\left. \frac{d}{dt} \right|_{t=0} f(\gamma(t)).$$

If γ_H denotes a horizontal lift of γ then this derivative is also equal to

$$\left. \frac{d}{dt} \right|_{t=0} \hat{f}(\gamma_H(t)).$$

where $\hat{f}(p) = f(\pi(p))$ ($\pi: L \rightarrow M$ denotes the projection). The point of this discussion is that we can identify $\mathcal{L}_{(\hat{\xi})_H} A(\widehat{d\sigma(\eta)})$ with

$$\mathcal{L}_{\hat{\xi}} A(\widehat{d\sigma(\eta)}) = \hat{\xi} \cdot A(\widehat{d\sigma(\eta)})$$

where $\xi \cdot f$ denotes the action of $\xi \in \mathfrak{g}$ on a function f in the \mathfrak{g} -module $\text{Map}(M, \mathbb{C})$. From here it is easy to see that $b(\xi) = A(\overline{d\sigma(\xi)})$ is the required coboundary.

For the remainder of the paper we would like to study the example of this general situation mentioned in the introduction; namely the extension of groups described in [8] associated to a the determinant line bundle Det_2 over the infinite dimensional Grassmannian manifold Gr_2 . In the sections that follow we briefly review the construction of this determinant line bundle and discuss its geometry.

3 The geometry of the Grassmannians Gr_p

The description of the determinant line bundle Det_p requires a basic knowledge of the Schatten ideals L_p in the algebra of bounded operators $\mathcal{B}(H)$ on a separable complex Hilbert space H . Briefly L_p is defined to be the set of all operators $A \in \mathcal{B}(H)$ such that $\text{tr}(AA^*)^{p/2} < \infty$. If $A \in L_p$ then we write $\|A\|_p = (\text{tr}(AA^*)^{p/2})^{1/p}$. It can be shown that $\|\cdot\|_p$ defines a norm on L_p and that L_p is an ideal in $\mathcal{B}(H)$ for any $p \geq 1$. As remarked earlier in the introduction, L_1 consists of the trace class operators on H and L_2 consists of the Hilbert-Schmidt operators. The ideals L_p share many of the properties of the measure spaces $L^p(X)$; for example if $A \in L_q$, $B \in L_r$ and $p^{-1} = q^{-1} + r^{-1}$ then $AB \in L_p$ and $\|AB\|_p \leq \|A\|_q \|B\|_r$. For more details the reader should consult [14].

To describe the line bundles Det_p we first need to describe the manifolds Gr_p over which they are defined. Gr_p is an infinite dimensional Grassmannian manifold associated to a complex, infinite dimensional, separable Hilbert space H which is equipped with a polarization $H = H_+ \oplus H_-$. To this polarization we can associate the operator

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is 1 on the subspace H_+ and -1 on the subspace H_- . ϵ is self adjoint and $\epsilon^2 = 1$. In terms of the projections pr_+ and pr_- onto the subspaces H_+ and H_- respectively, ϵ can be written as $\epsilon = 2\text{pr}_+ - 1$. This operator ϵ plays a useful role in defining the manifold Gr_p as we now explain.

In the case of the ordinary Grassmannian $\text{Gr}(V)$ associated to a vector space V , there are several different ways to describe points in $\text{Gr}(V)$. These different ways are described in [12]. One can either think of a point of $\text{Gr}(V)$ as subspace $W \subset V$, or equivalently we can replace the subspace W with the orthogonal projection P_W onto it. We can replace the orthogonal projection P_W with the self adjoint involution F of V defined by $F = 2P_W - 1$. Clearly we can move back and forth between projections and involutions this way. Finally, the group $GL(V)$ acts transitively on $\text{Gr}(V)$, and this leads to another description of $\text{Gr}(V)$ as a homogenous space. These four descriptions of points in Grassmannians persist to the infinite dimensional case of Gr_p . From [8, 12] we have the following descriptions of points in Gr_p :

1. a point of Gr_p can be thought of as a subspace $W \subset H$ such that the orthogonal projections $\text{pr}_+ : W \rightarrow H_+$ and $\text{pr}_- : W \rightarrow H_-$ are Fredholm and L_{2p} operators respectively,
2. a point of Gr_p can be thought of as a self adjoint projection P on H such that $[P, \epsilon] \in L_{2p}$,
3. a point of Gr_p can be thought of as a self adjoint bounded operator F on H such that $F - \epsilon \in L_p$ and $F^2 = 1$,
4. Gr_p can be thought of as the homogenous space $GL_{(p)}/B_{(p)}$ where $B_{(p)}$ is the subgroup of $GL_{(p)}$ consisting of invertible operators with block diagonal decomposition of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Each of these definitions have their own advantages, for instance from 4 it is clear that $\text{Gr}_{(p)}$ has a natural structure as a complex Banach manifold. We find the description in 3 in terms of involutions the most convenient for our purposes. For the remainder of this paper we shall only be interested in the L_4 Grassmannian Gr_2 and we will take a moment to amplify the description in 3 for this case. Points in the L_4 Grassmannian are self adjoint bounded involutions F on H such that $F - \epsilon \in L_4$. With respect to the polarization $H = H_+ \oplus H_-$ we can write

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

Thus F_{11} and F_{22} are self adjoint and $F_{21} = F_{12}^*$. Since $F - \epsilon \in L_4$, we must have $(F - \epsilon)^2 \in L_2$. From here we see that $F_{11} - 1 \in L_2$ and $F_{22} + 1 \in L_2$. Since $F^2 = 1$ and $F_{11} \in 1 + L_2$ we see also that $F_{12}F_{12}^* \in L_2$ and hence $F_{12} \in L_4$.

Associated to each such involution F is a subspace W satisfying the conditions of 1 above. In particular the orthogonal projection $\text{pr}_+ : W \rightarrow H_+$ is a Fredholm operator, the index of which is called the *virtual dimension* of W (see [11]). In general there are many components of Gr_2 and in fact these components are labelled by the virtual dimension. In this paper we will just be concerned with the connected component $(\text{Gr}_2)_0$ of Gr_2 consisting of planes W of virtual dimension zero. For this reason we will indulge in a slight abuse of notation and write Gr_2 when we really mean $(\text{Gr}_2)_0$.

Over Gr_2 there is defined (see [6, 8, 11]) a ‘Steifel’ bundle $\text{St}_2 \rightarrow \text{Gr}_2$ of ‘admissible frames’. This is a principal bundle with structure group GL^2 , the invertible operators g on H_+ such that $g - 1$ is Hilbert-Schmidt. The space St_2 can be described by

$$\text{St}_2 = \{w : H_+ \rightarrow H \mid w \text{ is injective, } \text{pr}_+ w - 1 \in L_2, \text{pr}_- w \in L_4\}.$$

If $w \in \text{St}_2$ then the image $w(H_+)$ is a subspace W of H satisfying the conditions of 1 above, and hence W defines a point of Gr_2 . The invertible operator

$w: H_+ \rightarrow H$ defines a basis for W ; this is an admissible frame in the sense of [6, 11]. Thus St_2 is an open subset of the Banach space which is the subspace of $\mathcal{B}(H_+, H)$ consisting of all bounded operators $T: H_+ \rightarrow H$ such that $\text{pr}_+ T - 1 \in L_2$ and $\text{pr}_- T \in L_4$. We equip this subspace with the topology coming from the metric

$$\|T - T'\| = \|\text{pr}_+ T - \text{pr}_+ T'\|_2 + \|\text{pr}_- T - \text{pr}_- T'\|_4$$

where $\|\cdot\|_2$ and $\|\cdot\|_4$ denote the norms of the Banach spaces L_2 and L_4 respectively. St_2 has a natural structure of a Banach manifold, since it is an open subset of a Banach space. The projection $\text{St}_2 \rightarrow \text{Gr}_2$ sends an admissible frame w to the orthogonal projection P_W onto its image W (P_W is identified with an involution in the usual way).

Suppose that $X: H_+ \rightarrow H$ is a linear map such that $\text{pr}_+ X \in L_2$ and $\text{pr}_- X \in L_4$. If $w \in \text{St}_2$ then $w + tX$ is an injective map if the real number t is small enough. Therefore we see that the tangent space to St_2 at w can be identified with the Banach space of all $X \in \mathcal{B}(H_+, H)$ with $\text{pr}_+ X \in L_2$ and $\text{pr}_- X \in L_4$ described above.

There is a natural connection 1-form Θ on the principal GL^2 bundle $\text{St}_2 \rightarrow \text{Gr}_2$ defined by

$$\Theta = w^{-1} P_W dw$$

Clearly this is Ad-invariant and restricts to the Maurer-Cartan form on each fibre. In order to be a connection 1-form however Θ needs to take values in the Hilbert-Schmidt operators on H_+ . To see that this is the case observe that

$$w^{-1} P_W dw = (\text{pr}_+ w)^{-1} \text{pr}_+ P_W dw + (\text{pr}_- w)^{-1} \text{pr}_- P_W dw.$$

The following Lemma is a standard calculation.

Lemma 2. *The curvature 2-form Ω of the connection Θ is given by*

$$\Omega = w^{-1} P_W dP_W dP_W w.$$

4 The regularized determinant line bundle Det_2

If g is a bounded operator which differs from the identity by a trace class operator then we may form its determinant $\det(g)$. In fact this determinant operator restricts to a homomorphism of groups

$$\det: GL^1 \rightarrow \mathbb{C}^*$$

where the group GL^1 consists of invertible operators g such that $g - 1$ is trace-class. If the norm $|A|$ of A is sufficiently small then this determinant is defined by the usual formula

$$\det(1 + A) = \exp \left(\text{tr} \left(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} A^i \right) \right)$$

for trace-class operators A . This expression obviously has no meaning if $A \in L_2$. However, one can regularize this determinant by removing the divergent part of the trace and defining

$$\det_2(1 + A) = \exp \left(\text{tr} \left(\sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{i} A^i \right) \right)$$

for $A \in L_2$ close to zero. In general, to define \det_2 we proceed as follows (see [14]). For any bounded operator A and any positive integer n , define

$$\mathcal{R}_n(A) = (1 + A) \exp \left(\sum_{j=1}^{n-1} (-1)^j \frac{A^j}{j} \right) - 1$$

It is shown in Lemma 9.1 of [14] that if $A \in L_n$, then $\mathcal{R}_n(A) \in L_1$. One then defines [14], for any positive integer n and $A \in L_n$,

$$\det_n(1 + A) = \det(1 + \mathcal{R}_n(A)).$$

When $n = 2$ this process defines a map

$$\det_2: GL^2 \rightarrow \mathbb{C}^*,$$

however this is not a homomorphism; instead we have (see [14])

$$\det_2(1 + A)(1 + B) = \det_2(1 + A)\det_2(1 + B) \exp(-\text{tr}(AB)) \quad (12)$$

for $A, B \in L_2$. Even though the regularized determinant $\det_2: GL^2 \rightarrow \mathbb{C}^*$ is not a homomorphism, we can still use it to define a determinant line bundle Det_2 on Gr_2 as follows. Following [8] we define an action of GL^2 on the quotient $\text{St}_2 \times \mathbb{C}^*$ by

$$(w, z)g = (wg, z\omega(w_+, g)^{-1})$$

where $w_+ = \text{pr}_+ w$. Here $\omega(w_+, g)$ is the function defined by

$$\omega(w_+, g) = \det_2(g) \exp(-\text{tr}(w_+ - 1)(g - 1))$$

Then, as in [8], we let Det_2 denote the quotient space $\text{Det}_2 = (\text{St}_2 \times \mathbb{C}^*)/GL^2$. It can be shown that Det_2 is a smooth principal \mathbb{C}^* bundle on Gr_2 . In fact, since Det_2 is a quotient of two complex manifolds, it is a holomorphic line bundle.

5 A connection 1-form on Det_2

A good way to define a connection 1-form on Det_2 is to regard Det_2 as an example of what Michael Murray has called a *pre-line bundle* [2]. A pre-line bundle on a manifold M consists of a surjective submersion $\pi: Y \rightarrow M$ together with a smooth map $f: Y^{[2]} \rightarrow \mathbb{C}^*$ satisfying the ‘cocycle condition’

$$f(y_2, y_3)f(y_1, y_3)^{-1}f(y_1, y_2) = 1$$

for points y_1, y_2, y_3 all lying in the same fibre of Y over M . Here $Y^{[2]} = \{(y_1, y_2) \mid \pi(y_1) = \pi(y_2)\}$. It is a smooth submanifold of Y^2 . Given a pre-line bundle (Y, f) on M we can construct a principal \mathbb{C}^* bundle (and hence an associated line bundle) by forming the product $Y \times \mathbb{C}^*$ and introducing the equivalence relation which identifies

$$(y, z) \sim (y', zf(y, y'))$$

for $(y, y') \in Y^{[2]}$. The quotient of $Y \times \mathbb{C}^*$ by this equivalence relation defines a principal \mathbb{C}^* bundle P over M , and in fact every principal \mathbb{C}^* bundle on M arises in this way. One can construct a connection 1-form on this bundle as follows. Suppose that there is a 1-form A on Y such that

$$f^{-1}df = \pi_2^*A - \pi_1^*A \quad (13)$$

where $\pi_1, \pi_2: Y^{[2]} \rightarrow Y$ denote the maps which omit the first and second factors in $Y^{[2]}$ respectively (such a 1-form A will exist if M admits partitions of unity). Given a 1-form A satisfying (13), the 1-form

$$A + z^{-1}dz \quad (14)$$

on $Y \times \mathbb{C}^*$ descends to the quotient and defines a connection 1-form on the principal \mathbb{C}^* bundle P . Since pre-line bundles are not a familiar notion we will include the details of this. We will show that the 1-form (14) descends to a 1-form on P . Suppose that (y_1, z_1) and (y_2, z_2) are points lying in the same fiber of $Y \times \mathbb{C}^*$ over P , and $(Y_1, \alpha_1), (Y_2, \alpha_2)$ are tangent vectors at $(y_1, z_1), (y_2, z_2)$ respectively which pushforward to the same tangent vector on P . We need to show that

$$A(Y_1) + \alpha_1 = A(Y_2) + \alpha_2.$$

Since (y_1, z_1) and (y_2, z_2) lie in the same fiber over P we must have $\pi(y_1) = \pi(y_2)$ and $z_2 = z_1 f(y_1, y_2)$. Similarly we must have $\alpha_2 = \alpha_1 + f^{-1}df(Y_1, Y_2)$. Therefore

$$A(Y_2) + \alpha_2 = A(Y_2) + \alpha_1 + f^{-1}df(Y_1, Y_2) = A(Y_1) + \alpha_1$$

as required. It is easy to show that push forward of (14) is invariant and restricts to the Maurer-Cartan 1-form on the fibers.

As remarked above, the principal \mathbb{C}^* bundle Det_2 is an example of a pre-line bundle for the submersion $\text{St}_2 \rightarrow \text{Gr}_2$. The cocycle f in this case is defined to be

$$f(w_1, w_2) = \omega((w_1)_+, g)^{-1}$$

for $(w_1, w_2) \in \text{St}_2^{[2]}$ and where g is the unique element of GL^2 such that $w_2 = w_1 g$. The cocycle condition $f(w_2, w_3)f(w_1, w_3)^{-1}f(w_1, w_2) = 1$ is easy to check. As mentioned in the introduction, it is possible to define a Hermitian structure on Det_2 (see [8]). As a Hermitian holomorphic line bundle Det_2 therefore has a canonical connection, however obtaining a closed formula for its curvature

seems to be rather difficult. Therefore we have constructed a connection on Det_2 using the theory of pre-line bundles, as we now explain.

We need to calculate $f^{-1}df$. This is a somewhat longwinded calculation, and for this reason we have relegated it to Appendix B. The result is that $f^{-1}df$ is equal to

$$\text{tr}((w_2)_+ - 1)\pi_1^*\Theta - ((w_1)_+ - 1)\pi_2^*\Theta - \text{pr}_+\pi_2^*\text{pr}_{H_w}dw + \text{pr}_+\pi_1^*\text{pr}_{H_w}dw \quad (15)$$

Here Θ is an arbitrary connection 1-form on the principal bundle $\text{St}_2 \rightarrow \text{Gr}_2$, and $\text{pr}_{H_w}dw$ denotes the operator valued 1-form on St_2 defined by the orthogonal projection of $dw: T\text{St}_2 \rightarrow \mathcal{B}(H_+, H)$ onto the horizontal subspace (with respect to Θ) at w . In order to write this as $\pi_2^*A - \pi_1^*A$ for some 1-form A on St_2 we need a connection Θ on the principal GL^2 bundle $\text{St}_2 \rightarrow \text{Gr}_2$ such that $\text{pr}_+\text{pr}_{H_w}dw$ is trace class. Such a connection is given for example by

$$\Theta = w^{-1}P_Wdw - w^{-1}P_W\text{pr}_+dP_{W^\perp}w = w^{-1}P_Wdw + w^{-1}P_W\text{pr}_+P_{W^\perp}dw. \quad (16)$$

To see this note that first of all $\text{pr}_+dP_{W^\perp}w = \text{pr}_+dP_{W^\perp}\text{pr}_+w + \text{pr}_+dP_{W^\perp}\text{pr}_-w$ takes values in L_2 , since $\text{pr}_+dP_{W^\perp}\text{pr}_+ \in L_2$ and $\text{pr}_+dP_{W^\perp}\text{pr}_-, \text{pr}_-w \in L_4$. Also, for this choice of Θ note that if X is a tangent vector at w then the horizontal projection $\text{pr}_{H_w}X$ onto H_w is given by

$$X - ww^{-1}P_WX - ww^{-1}P_W\text{pr}_+P_{W^\perp}X = P_{W^\perp}X - P_W\text{pr}_+P_{W^\perp}X.$$

Therefore

$$\begin{aligned} \text{pr}_+\text{pr}_{H_w}X &= \text{pr}_+P_{W^\perp}X - \text{pr}_+P_W\text{pr}_+P_{W^\perp}X \\ &= \text{pr}_+P_{W^\perp}\text{pr}_+P_{W^\perp}X \\ &= \left(\frac{1-F_{11}}{2}\right)\left(\frac{1-F_{11}}{2}\right)\text{pr}_+X + \left(\frac{1-F_{11}}{2}\right)\left(\frac{-F_{12}}{2}\right)\text{pr}_-X \end{aligned}$$

Since $F_{11} - 1 \in L_2$ we see that this takes trace class values. Finally then we can write down a connection 1-form on Det_2 . We describe this connection 1-form in the following proposition, where we also give an explicit formula for its curvature.²

Proposition 3. *A connection 1-form on the principal \mathbb{C}^* -bundle $\text{Det}_2 \rightarrow \text{Gr}_2$ is given by*

$$-\text{tr}(\text{pr}_+dw - w^{-1}P_Wdw - w^{-1}P_W\text{pr}_+P_{W^\perp}dw) + z^{-1}dz.$$

The curvature of this connection 1-form is the 2-form on Gr_2 defined by

$$-\frac{1}{16}\text{tr}((F - \epsilon)^2 F dF dF)$$

²Jouko Mickelsson has informed me that he knew this expression for the first Chern class of Gr_2 , in a slightly different, but equivalent form.

From equation (15) we see that for the choice (16) of the connection Θ on St_2 , the following 1-form A on $\text{St}_2 \times \mathbb{C}^*$ satisfies the equation $\pi_2^* A - \pi_1^* A = f^{-1} df$:

$$A = -\text{tr}((w_+ - 1)\Theta + \text{pr}_+ \text{pr}_{H_w} dw).$$

Therefore, by the general principles of pre-line bundles described above a connection 1-form for Det_2 is given by the push forward of the 1-form

$$-\text{tr}((w_+ - 1)\Theta + \text{pr}_+ \text{pr}_{H_w} dw) + z^{-1} dz.$$

It is an easy calculation, using the definition of Θ , to see that this is equal to

$$\begin{aligned} & -\text{tr}((w_+ - 1)(w^{-1} P_W dw + w^{-1} P_W \text{pr}_+ P_{W^\perp} dw) \\ & \quad + \text{pr}_+ P_{W^\perp} \text{pr}_+ P_{W^\perp} dw) + z^{-1} dz \end{aligned}$$

By straightforward manipulations one can show that this expression is the same as the one in the Proposition. To find the curvature we need to find d of the 1-form $-\text{tr}(\text{pr}_+ dw - w^{-1} P_W dw - w^{-1} P_W \text{pr}_+ P_{W^\perp} dw)$. In order to do this we will make use of the fact that $ww^{-1} P_W = P_W$, which gives on differentiation $d(w^{-1} P_W) = w^{-1} P_W dP_W - w^{-1} P_W dw w^{-1} P_W$. We calculate

$$\begin{aligned} & -\text{tr}(-w^{-1} P_W dP_W dw + w^{-1} P_W dw w^{-1} P_W dw - w^{-1} P_W dP_W \text{pr}_+ P_{W^\perp} dw \\ & \quad + w^{-1} P_W dw w^{-1} P_W \text{pr}_+ P_{W^\perp} dw + w^{-1} P_W \text{pr}_+ dP_W dw) \end{aligned}$$

since $dP_{W^\perp} = -dP_W$. The term $w^{-1} P_W dw w^{-1} P_W dw$ is trace class and its trace is easily seen to vanish due to the following property³ of the operator trace: if A and B are Hilbert-Schmidt operators then $\text{tr}(AB) = \text{tr}(BA)$. Also, we can differentiate the identity $P_W w = w$ to obtain $dw = dP_W w + P_W dw$, and using this we can re-write the term $w^{-1} P_W dP_W dw$ as

$$\begin{aligned} w^{-1} P_W dP_W dw &= w^{-1} P_W dP_W dP_W w + w^{-1} P_W dP_W P_W dw \\ &= w^{-1} P_W dP_W dP_W w, \end{aligned}$$

where we have used the fact that $P_W dP_W P_W = 0$. Finally, we can observe that the term $w^{-1} P_W dw w^{-1} P_W \text{pr}_+ P_{W^\perp} dw = w^{-1} P_W dw w^{-1} P_W \text{pr}_+ dP_W w$ is trace class and, using the cyclic property of the trace mentioned above, we can write

$$\text{tr}(w^{-1} P_W dw w^{-1} P_W \text{pr}_+ dP_W w) = \text{tr}(w^{-1} P_W \text{pr}_+ dP_W P_W dw).$$

Therefore our expression for the curvature becomes

$$\begin{aligned} & -\text{tr}(-w^{-1} P_W dP_W dP_W w - w^{-1} P_W dP_W \text{pr}_+ P_{W^\perp} dw \\ & \quad - w^{-1} P_W \text{pr}_+ dP_W P_W dw + w^{-1} P_W \text{pr}_+ dP_W dw) \end{aligned}$$

³ More generally (see [14] Corollary 3.8) if $A, B \in \mathcal{B}(H)$ have the property that $AB \in L_1$ and $BA \in L_1$, then $\text{tr}(AB) = \text{tr}(BA)$.

Using the identity $dw = dP_W w + P_W dw$ we can simplify the last two terms in the above expression to $w^{-1}P_W \text{pr}_+ dP_W dP_W w$. Thus our new expression for the curvature is

$$-\text{tr}(-w^{-1}P_W \text{pr}_- dP_W dP_W w - w^{-1}P_W dP_W \text{pr}_+ dP_W w).$$

Each of the terms inside the trace belongs to the trace class ideal. Since $dP_W dP_W P_W = P_W dP_W dP_W$ and $dP_W P_W dP_W = P_W^\perp dP_W dP_W$ it is not hard to see that we can re-write the above expression as

$$-\text{tr}(-\text{pr}_- P_W dP_W dP_W + \text{pr}_+ P_W^\perp dP_W dP_W)$$

Again, each of the expressions $\text{pr}_- P_W dP_W dP_W$ and $\text{pr}_+ P_W^\perp dP_W dP_W$ is trace class and so we may finally re-write this expression as

$$-\text{tr}(-P_W \text{pr}_+ P_W dP_W dP_W + P_W^\perp \text{pr}_- P_W^\perp dP_W dP_W)$$

In terms of the involutions F associated to the projections P_W , this expression becomes

$$-\frac{1}{16}\text{tr}((F - \epsilon)^2 F dF dF).$$

The extra factors of $F - \epsilon$ serve to regularize the trace of the usual curvature 2-form $F dF dF$ of the finite dimensional Grassmannian. As a consistency check one can also see that this gives a closed form.

6 The Mickelsson-Rajeev cocycle

In [8] the Lie algebra 2-cocycle ω_A associated to the abelian extension $1 \rightarrow \text{Map}(\text{Gr}_2, \mathbb{C}^*) \rightarrow \widehat{GL}_{(2)} \rightarrow GL_{(2)} \rightarrow 1$ was computed. This is the well-known *Mickelsson-Rajeev cocycle*

$$\omega_A = \frac{1}{8}\text{tr}_C[[\epsilon, X], [\epsilon, Y]](\epsilon - F)$$

where tr_C denotes the *conditional trace*. Recall that tr_C is a regularization of the ordinary operator trace tr defined for operators A on a polarized Hilbert space $H = H_+ \oplus H_-$ by

$$\text{tr}_C(A) = \frac{1}{2}\text{tr}(A + \epsilon A \epsilon)$$

whenever the latter trace exists. An operator A for which $\text{tr}_C(A)$ is defined is called conditionally trace class. Clearly every trace class operator A is conditionally trace class and moreover $\text{tr}(A) = \text{tr}_C(A)$ in this case. It is an easy calculation to show that ω_A may be re-written in terms of the usual operator trace as

$$\omega_A = \frac{1}{16}\text{tr}(F - \epsilon)^2 \epsilon [[\epsilon, X], [\epsilon, Y]] \quad (17)$$

We will now compare ω_A to the cocycle ω_G defined using the connection described in the preceding section. In order to do this we need to compute the fundamental vector field \hat{X} on Gr_2 associated to the infinitesimal action of a vector $X \in \mathfrak{g}_{(2)}$. Since $g \in GL_{(2)}$ acts on $F \in \text{Gr}_2$ by $g(F) = gFg^{-1}$ we see that $\hat{X}_F = [F, X]$. Therefore the cocycle ω_G is given by

$$\omega_G = \frac{1}{16} \text{tr}(F - \epsilon)^2 F [[F, X], [F, Y]] \quad (18)$$

Although there are similarities between the two cocycles (17) and (18), it seems as least as hard to guess a coboundary relating them as to use the formula (11) of Proposition 1. In Appendix A we use the latter method to derive the following expression for a coboundary b relating the two cocycles:

$$b(X)(F) = \frac{1}{16} \text{tr}((F - \epsilon)^3 (F + \epsilon) \epsilon [\epsilon, X]) - \frac{1}{16} \text{tr}((F - \epsilon)^4 \epsilon X). \quad (19)$$

A Calculating the coboundary

In this section we compute the coboundary

$$b(X) = A(\widehat{d\sigma(X)})$$

of Proposition 1 for the Mickelsson-Rajeev extension (3) when Det_2 is equipped with the connection 1-form defined in Proposition 3. For this we need to review some constructions from [8], in particular we need to understand the local structure of the group $\widehat{GL}_{(2)}$. Firstly, we denote by \mathcal{E}_2 the group

$$\mathcal{E}_2 = \{(g, q) \mid g \in GL_{(2)}, q \in GL(H_+), aq^{-1} - 1 \in L_2\}$$

where g is written in block diagonal form as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The product $\mathcal{E}_2 \times \text{Map}(\text{Gr}_2, \mathbb{C}^*)$ acts on the left of Det_2 through the formula

$$(g, q, f) \cdot (w, \lambda) = (gwq^{-1}, f(F)\lambda\alpha(g, q; w))$$

where F is the Hermitian involution corresponding to the plane W spanned by the admissible basis w , and where $\alpha(g, q; w)$ is a function satisfying a certain equivariance property. The function α is not unique, and Mickelsson and Rajeev make the choice

$$\alpha = \exp \left(-\text{tr}((1 - q^{-1}a)(w_+ - 1) + q^{-1}b(\frac{1}{2}F_{21} - w_-)) \right)$$

The group $\widehat{GL}_{(2)}$ is constructed from the product $\mathcal{E}_2 \times \text{Map}(\text{Gr}_2, \mathbb{C}^*)$ as a quotient:

$$(\mathcal{E}_2 \times \text{Map}(\text{Gr}_2, \mathbb{C}^*)) / N,$$

where N is a normal subgroup. The precise description of N will not be necessary for our calculation, and we refer to [8] for more details. In an open neighbourhood of the identity element of $GL_{(2)}$, Mickelsson and Rajeev define a local section $\sigma: GL_{(2)} \rightarrow \widehat{GL}_{(2)}$ by

$$\sigma(g) = (g, a, 1) \mod N.$$

Before we get to the calculation there is one last construction we need to review from [8]. Suppose that W is a subspace of H representing a point of Gr_2 and that $w: H_+ \rightarrow H$ is an admissible frame for W . Then we can define (see [8, 11]) an invertible operator

$$h = \begin{pmatrix} w_+ & \alpha \\ w_- & \beta \end{pmatrix}$$

such that $h(H_+) = W$ and $h(H_-) = W^\perp$. If we denote

$$h^{-1} = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

so that $x = w^{-1}P_W \text{pr}_+$ and $y = w^{-1}P_W \text{pr}_-$, then the involution F corresponding to W can be written $F = h\epsilon h^{-1}$. As noted in [8] this implies the pair of equations $F_{11} = 2w_+x - 1$ and $F_{21} = 2w_-x$.

We need to compute the fundamental vector field $\widehat{d\sigma(X)}$ where $d\sigma$ denotes the derivative of σ and X is a vector in $\mathfrak{gl}_{(2)}$. An easy computation gives that

$$\widehat{d\sigma(X)} = (Xw - wX_{11}, -\text{tr}(X_{12}(\frac{1}{2}F_{21} - w_-))).$$

We now need to compute the result of applying the connection 1-form A on Det_2 to this vector field. Note that A can be written as

$$-\text{tr}((1 - w^{-1}\text{pr}_W \text{pr}_+)\text{pr}_+ dw) + \text{tr}(w^{-1}P_W \text{pr}_- P_W dw) + z^{-1}dz$$

We first compute $\text{tr}(w^{-1}P_W \text{pr}_- P_W (Xw - wX_{11}))$. We can write this as

$$\text{tr}(w^{-1}P_W \text{pr}_- P_W X \text{pr}_+ w_+ + w^{-1}P_W \text{pr}_- P_W X \text{pr}_- w_- - w^{-1}P_W \text{pr}_- wX_{11}) \quad (20)$$

and then split (20) up into the following sum of traces (it is straightforward to check that all of the expressions involved are of trace class):

$$\begin{aligned} & \text{tr}(w^{-1}P_W \text{pr}_- P_W \text{pr}_- X_{21} w_+) + \text{tr}(w^{-1}P_W \text{pr}_- P_W X \text{pr}_- w_-) \\ & + \text{tr}(w^{-1}P_W \text{pr}_- P_W \text{pr}_+ X_{11} - w^{-1}P_W \text{pr}_- wX_{11}) \\ & + \text{tr}(w^{-1}P_W \text{pr}_- P_W \text{pr}_+ X_{11}(w_+ - 1)) \quad (21) \end{aligned}$$

By using the cyclic property of the trace we can further write (21) as

$$\begin{aligned} & \text{tr}(\text{pr}_+ P_W \text{pr}_- P_W \text{pr}_- X_{21} + \text{pr}_- P_W \text{pr}_- P_W X \text{pr}_- \\ & + (w_+ - 1)w^{-1}P_W \text{pr}_- P_W \text{pr}_+ X_{11}) \\ & + \text{tr}(w^{-1}P_W \text{pr}_- P_W \text{pr}_+ X_{11} - w^{-1}P_W \text{pr}_- wX_{11}) \quad (22) \end{aligned}$$

The term $\text{tr}((w_+ - 1)w^{-1}P_W\text{pr}_-P_W\text{pr}_+X_{11})$ in (22) can be written as

$$\text{tr}(\text{pr}_+P_W\text{pr}_-P_W\text{pr}_+X_{11} - w^{-1}P_W\text{pr}_-P_W\text{pr}_+X_{11}).$$

This can be combined with the last term in (22) to get

$$\text{tr}(\text{pr}_+P_W\text{pr}_-P_W\text{pr}_+X_{11} - w^{-1}P_W\text{pr}_-wX_{11}) = \text{tr}\left(\frac{F_{12}F_{21}}{4} - yw_-\right)X_{11}.$$

Therefore we find that we can write (22) as

$$\text{tr}(\text{pr}_+P_W\text{pr}_-P_W\text{pr}_-X_{21} + \text{pr}_-P_W\text{pr}_-P_WX\text{pr}_- + \left(\frac{F_{12}F_{21}}{4} - yw_-\right)X_{11}). \quad (23)$$

We now compute the contribution to $A(\widehat{d\sigma(X)})$ coming from

$$- \text{tr}((1 - w^{-1}P_W\text{pr}_+)(\text{pr}_+Xw - w_+X_{11})) \quad (24)$$

We write x for $w^{-1}P_W\text{pr}_+$. From $w_+x = \text{pr}_+P_W\text{pr}_+$ we see that $x \in 1 + L_2$. We also write $\text{pr}_+Xw = X_{11}w_+ + X_{12}w_-$. Then, after a little manipulation, (24) can be written as

$$- \text{tr}((xw_+ - w_+x)X_{11}) - \text{tr}((1 - x)X_{12}w_-) \quad (25)$$

Since $F_{21} = 2w_-x$ we see that we can write the term $\text{tr}(X_{12}(F_{21}/2 - w_-))$ as

$$\text{tr}(X_{12}(F_{21}/2 - w_-)) = -\text{tr}(X_{12}w_-(1 - x)) = -\text{tr}((1 - x)X_{12}w_-).$$

Therefore, on applying A to the fundamental vector field $\widehat{d\sigma(X)}$ we see that the contribution coming from $z^{-1}dz$ cancels with the term $-\text{tr}((1 - x)X_{12}w_-)$ appearing in the expression for $-\text{tr}(1 - w^{-1}P_W\text{pr}_+)(\text{pr}_+Xw - w_+X_{11})$. We obtain the following expression for $A(\widehat{d\sigma(X)})$:

$$\begin{aligned} & \text{tr}(\text{pr}_+P_W\text{pr}_-P_W\text{pr}_-X_{21} + \text{pr}_-P_W\text{pr}_-P_WX\text{pr}_-) \\ & + \text{tr}\left((w_+x - xw_+ - yw_- + \frac{F_{12}F_{21}}{4})X_{11}\right) \end{aligned}$$

Since $xw_+ + yw_- = 1$, the last term can be written as

$$\text{tr}\left((w_+x - 1 + \frac{F_{12}F_{21}}{4})X_{11}\right) = -\frac{1}{4}\text{tr}((1 - F_{11})^2X_{11}),$$

where we have used the identity $w_+x = (1 + F_{11})/2$. Therefore we have the following expression for the coboundary $b(X) = A(\widehat{d\sigma(X)})$, independent of w :

$$\text{tr}(\text{pr}_+P_W\text{pr}_-P_W\text{pr}_-X_{21} + \text{pr}_-P_W\text{pr}_-P_WX\text{pr}_-) - \frac{1}{4}\text{tr}((1 - F_{11})^2X_{11}).$$

We can further write the term $\text{tr}(\text{pr}_-P_W\text{pr}_-P_WX\text{pr}_-)$ as

$$\text{tr}(\text{pr}_-P_W\text{pr}_-P_W\text{pr}_+X_{12}) + \text{tr}(\text{pr}_-P_W\text{pr}_-P_W\text{pr}_-X).$$

Since $\text{pr}_- P_W \text{pr}_- = (F_{22} + 1)/2$ we can rewrite the formula for $b(X)$ as

$$\begin{aligned} & \text{tr}(\text{pr}_+ P_W \text{pr}_- P_W \text{pr}_- X_{21} + \text{pr}_- P_W \text{pr}_- P_W \text{pr}_+ X_{12}) \\ & + \frac{1}{4} \text{tr}((F_{22} + 1)^2 X) - \frac{1}{4} \text{tr}((F_{11} - 1)^2 X). \end{aligned}$$

We can write $b(X)$ in terms of F , ϵ and X by setting $\text{pr}_+ = \frac{1+\epsilon}{2}$, $P_W = \frac{1+F}{2}$ etc. For instance

$$F_{11} - 1 = \text{pr}_+(F - 1)\text{pr}_+ = \frac{1}{4}(1 + \epsilon)(F - 1)(1 + \epsilon) = -\frac{1}{4}(F - \epsilon)^2(1 + \epsilon)$$

using the identities $(F - \epsilon)^2 = 2 - \epsilon F - F\epsilon$ and $\epsilon F\epsilon = 2\epsilon - F - (F - \epsilon)^2\epsilon$. Similarly $F_{22} + 1 = (F - \epsilon)^2(1 - \epsilon)/4$. Thus the term $\frac{1}{4}\text{tr}((F_{22} + 1)^2 X) - \frac{1}{4}\text{tr}((F_{11} - 1)^2 X)$ above can be written as

$$-\frac{1}{16} \text{tr}((F - \epsilon)^4 \epsilon X).$$

Since $\text{pr}_- X_{21} = \text{pr}_-[\epsilon, X]$ and $\text{pr}_+ P_W \text{pr}_- = \frac{1}{8}(1 + \epsilon)F(1 - \epsilon)$ we can write

$$\begin{aligned} & \text{tr}(\text{pr}_+ P_W \text{pr}_- P_W \text{pr}_- X_{21}) \\ & = \frac{1}{32} \text{tr}((1 + \epsilon)F(F - \epsilon)^2(1 - \epsilon)[\epsilon, X]) \\ & = \frac{1}{32} \text{tr}((F - \epsilon)^2(1 + \epsilon)F(1 - \epsilon)[\epsilon, X]) \\ & = \frac{1}{32} \text{tr}((F - \epsilon)^3(F + \epsilon)(\epsilon - 1)[\epsilon, X]). \end{aligned}$$

We obtain a similar expression for $\text{tr}(\text{pr}_- P_W \text{pr}_- P_W \text{pr}_+ X_{12})$:

$$\text{tr}(\text{pr}_- P_W \text{pr}_- P_W \text{pr}_+ X_{12}) = \frac{1}{32} \text{tr}((F - \epsilon)^3(F + \epsilon)(\epsilon + 1)[\epsilon, X]).$$

Combining all of these expressions we can finally write

$$b(X) = \frac{1}{16} \text{tr}((F - \epsilon)^3(F + \epsilon)\epsilon[\epsilon, X]) - \frac{1}{16} \text{tr}((F - \epsilon)^4 \epsilon X).$$

B Derivation of equation (15)

Recall that $f: \text{St}_2^{[2]} \rightarrow GL^2$ is defined to be

$$f(w_1, w_2) = \omega((w_1)_+, g)^{-1} = (\det_2(g))^{-1} \exp(\text{tr}((w_1)_+ - 1)(g - 1)),$$

where $g = g(w_1, w_2)$ is the unique element of GL^2 such that $w_2 = w_1 g(w_1, w_2)$. Therefore $f^{-1}df$ is equal to

$$-(\det_2(g))^{-1} d\det_2(g) + \text{tr} \pi_2^* dw_+(g - 1) + \text{tr}((w_1)_+ - 1) dg \quad (26)$$

Thus we need to calculate the derivatives $d\det_2(g)$ and dg . The latter is easy, it is well known that

$$dg(X_1, X_2) = g\Theta(X_2) - \Theta(X_1)g \quad (27)$$

where (X_1, X_2) is a tangent vector at (w_1, w_2) in $\text{St}_2^{[2]}$ and Θ is an arbitrary connection 1-form on the principal GL^2 bundle $\text{St}_2 \rightarrow \text{Gr}_2$. To calculate $d\det_2(g)$ we need to calculate the derivative

$$\left. \frac{d}{dt} \right|_{t=0} \det_2(\exp(-t\Theta(X_1))g \exp(t\Theta(X_2))).$$

For this we will use the multiplicative property (12) of the regularized determinant \det_2 to write $\det_2(\exp(-t\Theta(X_1))g \exp(t\Theta(X_2)))$ as the product

$$\begin{aligned} & \det_2(\exp(-t\Theta(X_1))) \cdot \det_2(g) \cdot \det_2(\exp(t\Theta(X_2))) \cdot \\ & \quad \cdot \exp(-\text{tr}(g-1)(\exp(t\Theta(X_2)) - 1)) \cdot \\ & \quad \cdot \exp(-\text{tr}((\exp(-t\Theta(X_1)) - 1)(g \exp(t\Theta(X_2)) - 1)) \end{aligned}$$

We recall that $\det_2(\exp(tA))$ is $O(t^2)$ and so on taking derivatives we end up with

$$\det_2(g) (-\text{tr}((g-1)\Theta(X_2)) + \text{tr}(\Theta(X_1)(g-1))) \quad (28)$$

Combining (26), (27) and (28) we get the following formula for $f^{-1}df$:

$$\begin{aligned} & \text{tr}((g-1)\Theta(X_2)) - \text{tr}(\Theta(X_1)(g-1)) + \text{tr} \pi_2^* dw_+(g-1) \\ & \quad + \text{tr}((w_1)_+ - 1)(g\Theta(X_2) - \Theta(X_1)g) \end{aligned} \quad (29)$$

In this formula we can write $\pi_2^* dw_+ = (w_1)_+ A(X_1) + \pi_2^* \text{pr}_{H_w} dw$, where $\text{pr}_{H_w} dw$ denotes the operator valued form on St_2 which is the composition of the inclusion $dw: T\text{St}_2 \rightarrow \mathcal{B}(H_+, H)$ followed by orthogonal projection onto the horizontal subspace H_w defined by Θ . Then we see that (29) can be re-written as

$$\begin{aligned} & \text{tr}(((w_2)_+ - 1)\pi_1^* \Theta - ((w_1)_+ - 1)\pi_2^* \Theta - \text{pr}_+ \pi_2^* \text{pr}_{H_w} dw \\ & \quad + \pi_2^*(dw_+)g - (w_1)_+ \pi_2^* \Theta g) \end{aligned} \quad (30)$$

Since horizontal subspaces are translation invariant, we can write $\pi_2^*(dw)g - \pi_2^*(w\Theta)g = \pi_1^* \text{pr}_{H_w} dw$. Therefore (30) becomes

$$\text{tr}((w_2)_+ - 1)\pi_1^* \Theta - ((w_1)_+ - 1)\pi_2^* \Theta - \text{pr}_+ \pi_2^* \text{pr}_{H_w} dw + \text{pr}_+ \pi_1^* \text{pr}_{H_w} dw$$

which is equation (15).

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